AP Calculus AB
Test: Separable Differential Equations
Instructor's Version

Free-Response Question 1 — no calculator (15 points)

1. The equation $\frac{dy}{dx} = \frac{-6x^2}{y}$ gives the slope at any point on the graph of $f(x)$. The range of $f(x)$ is $[0, \infty]$ and $f(1) = 2$.

   A. Find the equation of the tangent line to $f(x)$ at the point (1, 2). (4 points)

   Answer:

   $\frac{dy}{dx} = \frac{-6(1)^2}{2} = -3$ (2 points: slope)

   $y - 2 = -3(x - 1)$ or $y = -3x + 5$ (2 points: answer)

   B. Write the function $f(x)$. (5 points)

   Answer:

   $\frac{dy}{dx} = \frac{-6x^2}{y} \xrightarrow{yields} ydy = -6x^2dx$ (1 point: separates variables)

   $\int ydy = \int -6x^2dx$ (1 point: integrates both sides)

   $\frac{1}{2}y^2 = -2x^3 + C$ (1 point: antiderivatives)

   $y^2 = -4x^3 + C$

   $(2)^2 = -4(1)^2 + C \xrightarrow{yields} C = 8$ (1 point: $C = 8$)

   $y = f(x) = \pm\sqrt{-4x^3 + 8}$

   $f(x) = \sqrt{-4x^3 + 8}$ (1 point: writes $f(x)$ to represent given range)

   C. Determine the domain of the function $f(x)$. (2 points)

   Answer:

   $-4x^3 + 8 \geq 0$ (1 point: inequality)

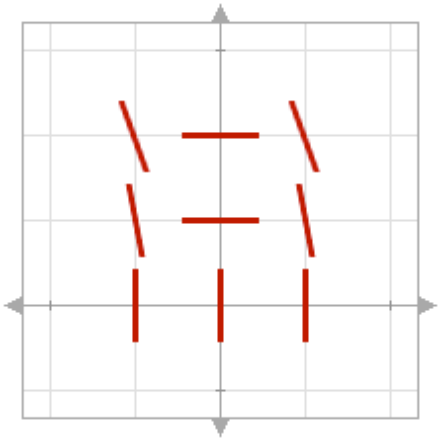
   $x^3 \leq 2$

   $x \leq \sqrt[3]{2}$ so domain is $\left(-\infty, \sqrt[3]{2}\right]$ (1 point: answer)

© Copyright 2009 Apex Learning Inc. All rights reserved. This material is intended for the exclusive use of registered users only. No portion of these materials may be reproduced or redistributed in any form without the express written permission of Apex Learning Inc.

D. Using the axes provided, sketch a slope field for the given differential equation at the nine points indicated. (4 points)

Answer:



1 point: zero slope at (0, 1) and (0, 2)
1 point: slope of -6 at (-1, 1) and (1, 1)
1 point: slope of -3 at (-1, 2) and (1, 2)
1 point: undefined slope along *x*-axis

© Copyright 2009 Apex Learning Inc. All rights reserved. This material is intended for the exclusive use of registered users only. No portion of these materials may be reproduced or redistributed in any form without the express written permission of Apex Learning Inc.

Free Response Question 2 — with calculator (15 points)

2. The rate of decay is proportional to the mass for radioactive material. For a certain radioactive isotope, this rate of decay is given by the differential equation $\frac{dm}{dt} = -.022m$, where *m* is the mass of the isotope in mg and t is the time in years.

   A. If *m*(0) = 20, write a function *m*(*t*) for the radioactive decay of the isotope. Show the steps in separating variables of the given differential equation and solving the equation for *m*. (4 points)

   Answer:

   $\int \frac{dm}{m} = -.022 \int dt$ (1 point: separates variables)

   $\ln|m| = -.022t + C$ (1 point: solves integral)

   $m = Ae^{-.022t}$

   $m(t) = 20e^{-.022t}$ (2 points: writes function)

   B. The half-life of a radioactive substance is the time required for half of the substance to decay. What is the half-life of this radioactive isotope to the nearest tenth of a year? (4 points)

   Answer:

   $\frac{1}{2} = e^{-.022t}$ (2 points: write equation)

   $t = \frac{\ln\left(\frac{1}{2}\right)}{-.022} \approx 31.5$ years (2 points: solve equation and include units)

   C. How many years (to the nearest tenth) will it take for the 20 mg to decay to 1 mg? (4 points)

   Answer:

   $\frac{1}{20} = e^{-.022t}$ (2 points: writes equation)

   $t = \frac{\ln\left(\frac{1}{20}\right)}{-.022} \approx 136.2$ years (2 points: answer with units)

© Copyright 2009 Apex Learning Inc. All rights reserved. This material is intended for the exclusive use of registered users only. No portion of these materials may be reproduced or redistributed in any form without the express written permission of Apex Learning Inc.

D. How many mg (to the nearest tenth) of the isotope will remain after 60 years? (3 points)

Answer:

$m(60) = 20e^{-.022*60}$ (1 point: writes equation)

$m(60) \approx 5.3$ mg (2 points: answer with units)

© Copyright 2009 Apex Learning Inc. All rights reserved. This material is intended for the exclusive use of registered users only. No portion of these materials may be reproduced or redistributed in any form without the express written permission of Apex Learning Inc.